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COMPUTATIONAL METHODS FOR ANTENNA PATTERN SYNTHESIS

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COMPUTATIONAL METHODS FOR ANTENNA PATTERN SYNTHESIS

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I. INTRODUCTION

The purpose of this report is to develop some general numerical methods for the problem of antenna pattern synthesis, with and without constraints. The principles of pattern synthesis are well known, and many specific problems have been considered [1]. A good discussion of the general theory and of the difficulties encountered has been given by Deschamps and Cabayan [2]. To overcome problems concerning the stability and sensitivity of the solution, they propose to use regularization methods [3]. In Hilbert space, this method is equivalent to obtaining a least-squares solution with a constraint on the source norm [4,5]. It is closely related to obtaining a least-squares solution with a constraint on the quality factor, as considered in Section VI of this report.

Most of the methods developed for pattern synthesis assume that the radiation field is specified in both magnitude and phase. In many cases only the magnitude is of interest, and the phase is left unspecified. This is a special case of the so-called mixed problems of antenna synthesis, considered by Bakhrakh and Troytskiy [6]. An application of field magnitude pattern

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[1] For example, see R. E. Collin and F. J. Zucker, "Antenna Theory," Part 1, McGraw-Hill Book Co., New York, 1969, Chap. 7.

[2] G. A. Deschamps and H. S. Cabayan, "Antenna Synthesis and Solution of Inverse Problems by Regularization Methods," IEEE Trans., vol. AP-20, No. 3, May 1972, pp. 268-274.

[3] A. N. Tihonov, "Solution of Incorrectly Formulated Problems and the Regularization Method," Soviet Mathematics, vol. 4, July-December 1963, pp. 1035-1038.

[4] V. I. Popovkin and V. I. Yelumeyev, "Optimization and Systematization of Solutions to Antenna Synthesis Problems," Radio Engineering and Electronic Physics, vol. 13, No. 5, 1968, pp. 682-686.

[5] V. I. Popovkin, G. I. Shcherbakov, V. I. Yelumeyev, "Optimum Solutions of Problems in Antenna Synthesis Theory," Radio Engineering and Electronic Physics, vol. 14, No. 7, 1969, pp. 1025-1030.

[6] L. D. Bakhrakh and V. I. Troytskiy, "Mixed Problems of Antenna Synthesis," Radio Engineering and Electronic Physics, vol. 12, No. 3, March 1967, pp. 404-414.

synthesis to a specific problem has been published by Choni [7]. In Section IV we give a discussion of field magnitude pattern synthesis in terms of numerical methods.

The procedures developed in this report are general, being applicable to any antenna system which can be accurately analyzed. However, to keep the examples and computer programs simple, the theory is applied explicitly only to point sources arbitrarily distributed in a plane, and to pattern synthesis in this plane. The extension to arbitrary N-port systems in three-dimensional space, and to pattern synthesis over the entire radiation sphere, is straightforward but tedious in detail. A special case of three-dimensional field magnitude pattern synthesis applied to wire scatterers is given in reference [8].

II. ANTENNA SYSTEMS AND VECTOR SPACES

The general relationship between the source f of a radiating system and the field g it produces on the radiation sphere can be symbolized by

$$Tf = g \tag{1}$$

where T is a linear operator. In general, both f and g may be infinite dimensional and T^{-1} may not be bounded. The pattern synthesis problem can be summarized as follows: Given a desired field g_o (specified completely or partially), we wish to determine a source f (constrained or unconstrained) whose field g approximates g_o in some acceptable manner. The approach taken by Deschamps and Cabayan [2] is to treat the problem in general in function spaces, and then numerically evaluate the final formulas for specific applications. Our approach is to discretize the problem at the beginning, and then treat the problem in finite-dimensional vector spaces.

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[7] Y. I. Choni, "Synthesis of an Antenna According to a Given Amplitude Radiation Pattern," Radio Engineering and Electronic Physics, vol. 16, No. 5, May 1971, pp. 770-778.

[8] R. F. Harrington and J. R. Mautz, "Synthesis of Loaded N-port Scatterers," AFCRL-72-0665, Scientific Report No. 17 on Contract No. F19628-68-C-0180 between Syracuse University and Air Force Cambridge Research Laboratories, October 1972.

To render the source discrete, we assume that

$$f = \sum_{n=1}^{N} f_n e_n \tag{2}$$

where f_n are constants and e_n are basis elements. If the source is continuous, the e_n are functions and (2) is usually an approximation to the true source. If the source is discrete, the e_n are finite-dimensional vectors, and (2) is an exact relationship. We define the vector $\vec{f}$ to be the vector of the components f_n, that is

$$\vec{f} = [f_n]_{N\times 1} \tag{3}$$

For example, if the source is an array of dipoles, the f_n may be the input port currents, or the input port voltages, or their components in some arbitrary basis. We substitute (2) into (1) and evaluate the equation at M points (θ_m,ϕ_m), $m=1,2,\ldots,M$ on the radiation sphere. The result can be written as the matrix equation

$$[T]\vec{f} = \vec{g} \tag{4}$$

where $\vec{g}$ is the vector

$$\vec{g} = [\vec{g}_m]_{M\times 1} \tag{5}$$

and [T] is the matrix

$$[T] = [(Te_n)_m]_{M\times N} \tag{6}$$

Here g_m denotes the value of g at the point (θ_m,ϕ_m), and $(Te_n)_m$ denotes the pattern of f_n evaluated at the point (θ_m,ϕ_m). If these quantities are spatial vectors, then the above procedure applies to the θ and ϕ components of the vector field. More generally, one could define a set of testing functions and reduce (1) to a matrix equation by the method of moments [9].

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[9] R. F. Harrington, "Field Computation by Moment Methods," Macmillan Co., New York, 1968.

The synthesis problem can now be represented by

$$[T]\vec{f} \approx \vec{g}_o \tag{7}$$

where $\vec{g}_o$ is the specified pattern vector. Equation (7) is a matrix equation, and the well-known methods of matrix algebra can now be brought to bear on the problem. If [T] is square and nonsingular, the solution is given by the inverse to (7). This solution will give the correct value of field at the M points (θ_m, ϕ_m) on the radiation sphere, but no control of the field is obtained between these points. If [T] is of rank less than N, the problem is underdetermined and more than one solution to (7) exists. In this case additional constraints can be applied to either the source or the field to obtain the most desirable solution. If [T] is of rank greater than N, then usually no exact solution to (7) exists, but there will be a unique least-squares solution. Again additional constraints can be applied to obtain more desirable solutions.

Let us next define the Euclidean spaces involved more precisely. For source quantities, the inner product is defined as

$$\langle f_i^*, f_j \rangle = \tilde{f}_i^*[V]\vec{f}_j \tag{8}$$

where the tilde denotes transpose, the asterisk denotes complex conjugate, and [V] is a weight matrix. [V] must be positive definite, and for this report we consider it to be the diagonal matrix

$$[V] = [\text{diag } v_n]_{N \times N} \tag{9}$$

with all $v_n > 0$. In the source space the norm induced by the inner product is

$$\|\vec{f}\| = \langle f^*, f \rangle^{1/2} = \left(\sum_{n=1}^{N} v_n |f_n|^2 \right)^{1/2} \tag{10}$$

where f_n are the components of $\vec{f}$. The metric induced by the norm is $d(\vec{f}_i, \vec{f}_j) = \|\vec{f}_i - \vec{f}_j\|$. For field quantities we define the inner product as

$$<g_i, g_j> = \tilde{\vec{g}}_i^{*}[W]\vec{g}_j \tag{11}$$

where [W] is a weight matrix. It must be positive definite and for this report we take it to be the diagonal matrix

$$[W] = [\text{diag } w_m]_{M \times M} \tag{12}$$

with all $w_m > 0$. In the field space the norm induced by the inner product is

$$||g|| = <g^*, g>^{1/2} = (\sum_{m=1}^{M} w_m |g_m|^2)^{1/2} \tag{13}$$

where g_m are the components of $\vec{g}$. The metric induced by the norm is $d(\vec{g}_i, \vec{g}_j) = ||\vec{g}_i - \vec{g}_j||$.

To compare various pattern synthesis procedures, we define two figures of merit. The first is the normalized synthesis error

$$E = \frac{||\vec{g} - \vec{g}_o||^2}{||\vec{g}_o||^2} = \frac{\sum_{m=1}^{M} w_m |g_m - g_{om}|^2}{\sum_{m=1}^{M} w_m |g_{om}|^2} \tag{14}$$

where $\vec{g}$ is the synthesized pattern and $\vec{g}_o$ is the desired pattern. The second is the quality factor [10]

$$Q = M \frac{||\vec{f}||^2}{||\vec{g}||^2} = M \frac{\sum_{n=1}^{N} v_n |f_n|^2}{\sum_{m=1}^{M} w_m |g_m|^2} \tag{15}$$

The multiplier M is introduced into (15) to make the Q relatively insensitive to the number of field points chosen.

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[10] Y. T. Lo, S. W. Lee, and Q. H. Lee, "Optimization of Directivity and Signal-to-Noise Ratio of an Arbitrary Antenna Array, " Proc. IEEE, vol. 54, August 1966, pp. 1033-1045.

III. PATTERN SYNTHESIS WITH PHASE SPECIFIED

The most commonly used method of pattern synthesis involves specifying the field pattern in both magnitude and phase. For our method, this involves specifying the magnitude and phase of g_o at M points on the radiation sphere. Hence, the starting point is (7) with $\vec{g}_o$ known. We normally take more independent equations than unknowns, in which case the least-squares solution to (7) is

$$\vec{f} = [\tilde{T}^{*}WT]^{-1}[\tilde{T}^{*}W]\vec{g}_o \tag{16}$$

This is obtained in the usual way by minimizing the quantity $\|[T]\vec{f}-\vec{g}_o\|^2$. The derivation of (16) follows (31) to (34) of Section IV with $\alpha=0$. If unweighted inner products are used, then [W] is the identity matrix.

For examples, the equations are specialized to point sources arbitrarily distributed in a plane and to radiation patterns in this same plane. Figure 1 illustrates the general problem. The point sources are located at positions (x_n,y_n) and have excitations f_n. The radiation pattern at the angle ϕ is then given by

$$Tf = \sum_{n=1}^{N} f_n e^{jk(x_n \cos\phi + y_n \sin\phi)} \tag{17}$$

where $k = 2\pi/\lambda$ is the wavenumber. Choosing a desired pattern $g_o(\phi)$, and setting $Tf \approx g_o$ at M points ϕ_m, we obtain $[T]\vec{f} \approx \vec{g}_o$. Here $\vec{f}$ is the column vector of the f_n, $\vec{g}_o$ is the column vector of the $g_o(\phi_m)$, and [T] is the matrix with elements

$$T_{mn} = e^{jk(x_n \cos\phi_m + y_n \sin\phi_m)} \tag{18}$$

For simplicity we take the unweighted inner product, in which case [W] is the identity matrix. The solution for the sources f_n is then given by (16), and the synthesized pattern by (17). This solution is valid so long as M is sufficiently large to give at least N independent equations in the N unknowns f_n.

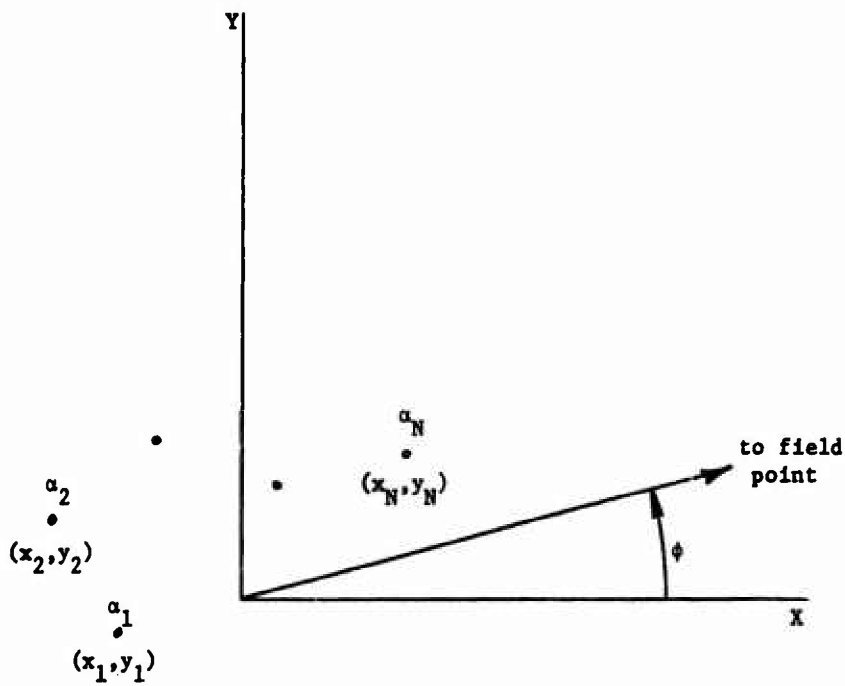
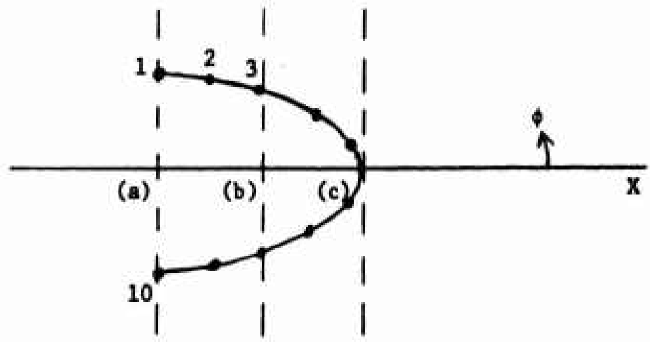


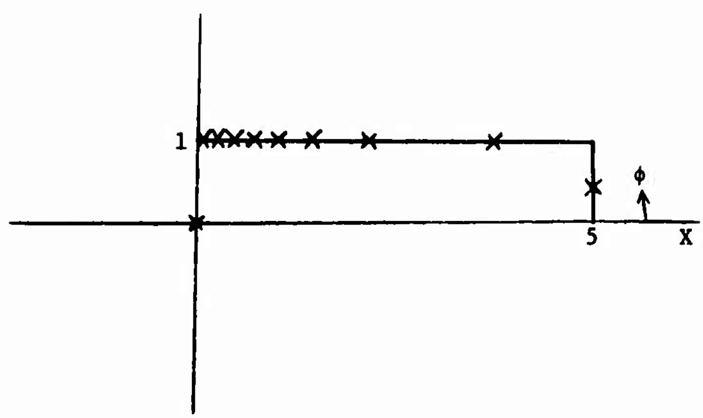
Fig. 1. An arbitrary array of point sources in the x-y plane.

For some numerical results, consider 10 sources approximately equispaced on one-half of a 2:1 ellipse, as shown in Fig. 2a. For the radiation pattern, take a rectangle in the first quadrant of the polar pattern. (This is the so-called cosecant ϕ pattern.) 36 field points ϕ_m are chosen every 10° in ϕ, starting at $\phi = 5°$. These are shown by crosses on Fig. 2b. For the synthesis we take four cases: (a) origin at the center of the ellipse and the field real, (b) origin one-half the distance from the center to the end point of the semimajor axis and the field real, (c) origin at the end point of the semimajor axis and the field real, and (d) origin at the center of the ellipse and the field phase alternating from 0 to 180° between adjacent field points. (This last case represents the worst choice that can be made.) Figure 3 shows the results when the sources are separated $d = \lambda/4$, Fig. 4 when $d = \lambda/2$, and Fig. 5 when $d = \lambda$. The corresponding normalized synthesis errors E and quality factors Q associated with each synthesis result are given below each pattern. The source excitations are listed by magnitude and phase (in degrees) in Tables 1 to 3, normalized so that the maximum excitation is unity.

Some general observations are as follows: When the field is chosen to have alternating phase between adjacent points (cases d), the pattern synthesis is poor in each case. When the field is chosen to be in phase at all points, with respect to a coordinate origin in the vicinity of the sources, the synthesis is good when the sources are separated by $d = \lambda/4$ and $\lambda/2$, and poor when separated by $d = \lambda$. The Q is low when the sources are separated by $d = \lambda/2$ and λ, but relatively high when $d = \lambda/4$. These properties are in agreement with what we would expect based on past experience with antenna synthesis problems.

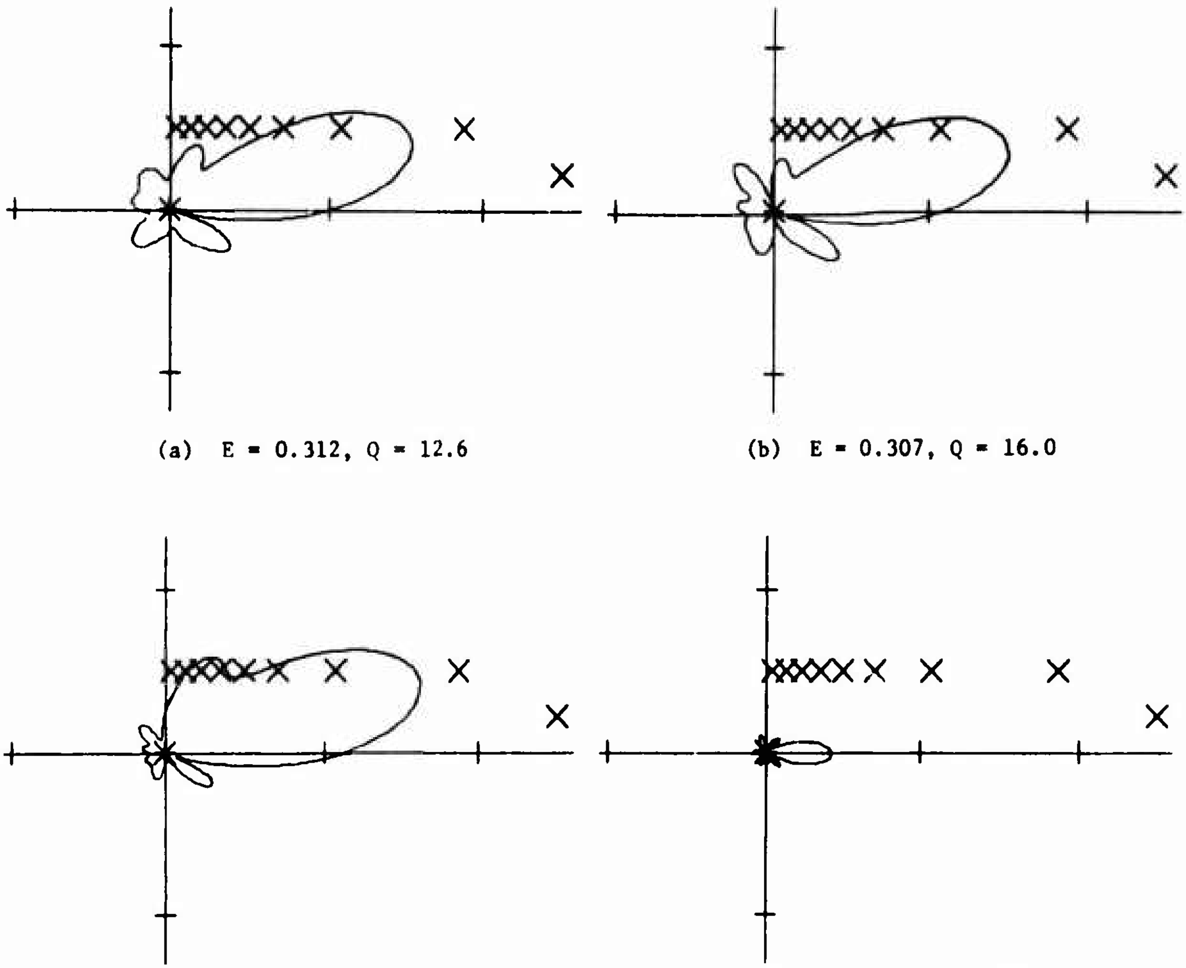


(a) Source distribution



(b) Desired pattern

Fig. 2. Source distribution and desired pattern for numerical example.



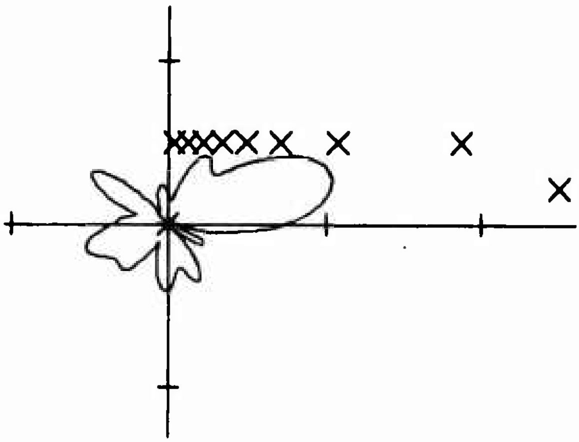
(a) E = 0.312, Q = 12.6

(b) E = 0.307, Q = 16.0

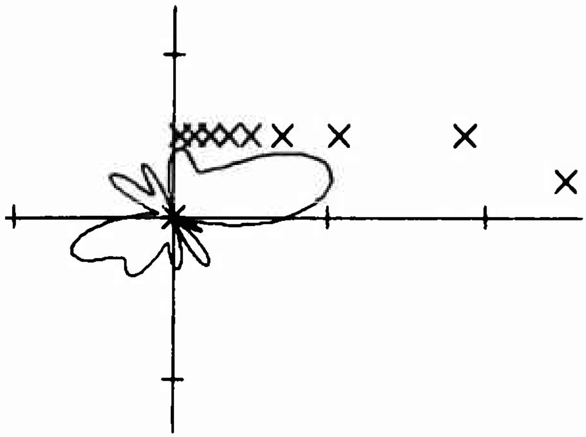
(c) E = 0.223, Q = 11.1

(d) E = 0.957, Q = 20.5

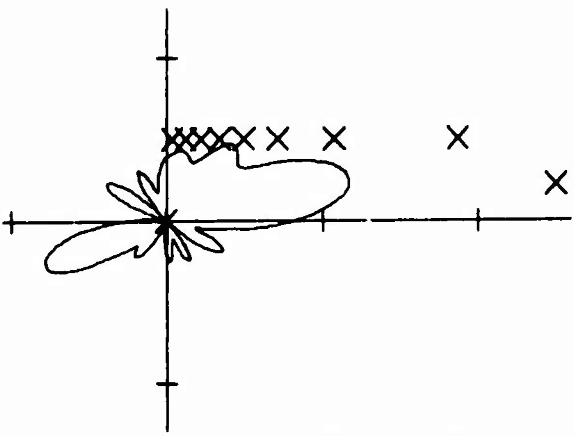
Fig. 3. Antenna pattern synthesis, phase specified, d = $\lambda/4$. Specified pattern shown by crosses, synthesized pattern shown solid.



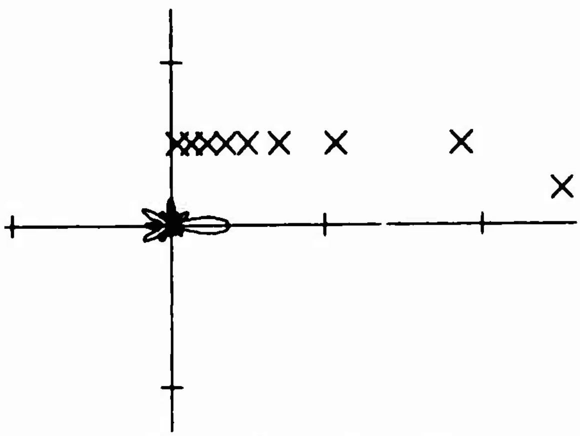
(a) E = 0.494, Q = 1.86



(b) E = 0.495, Q = 1.70

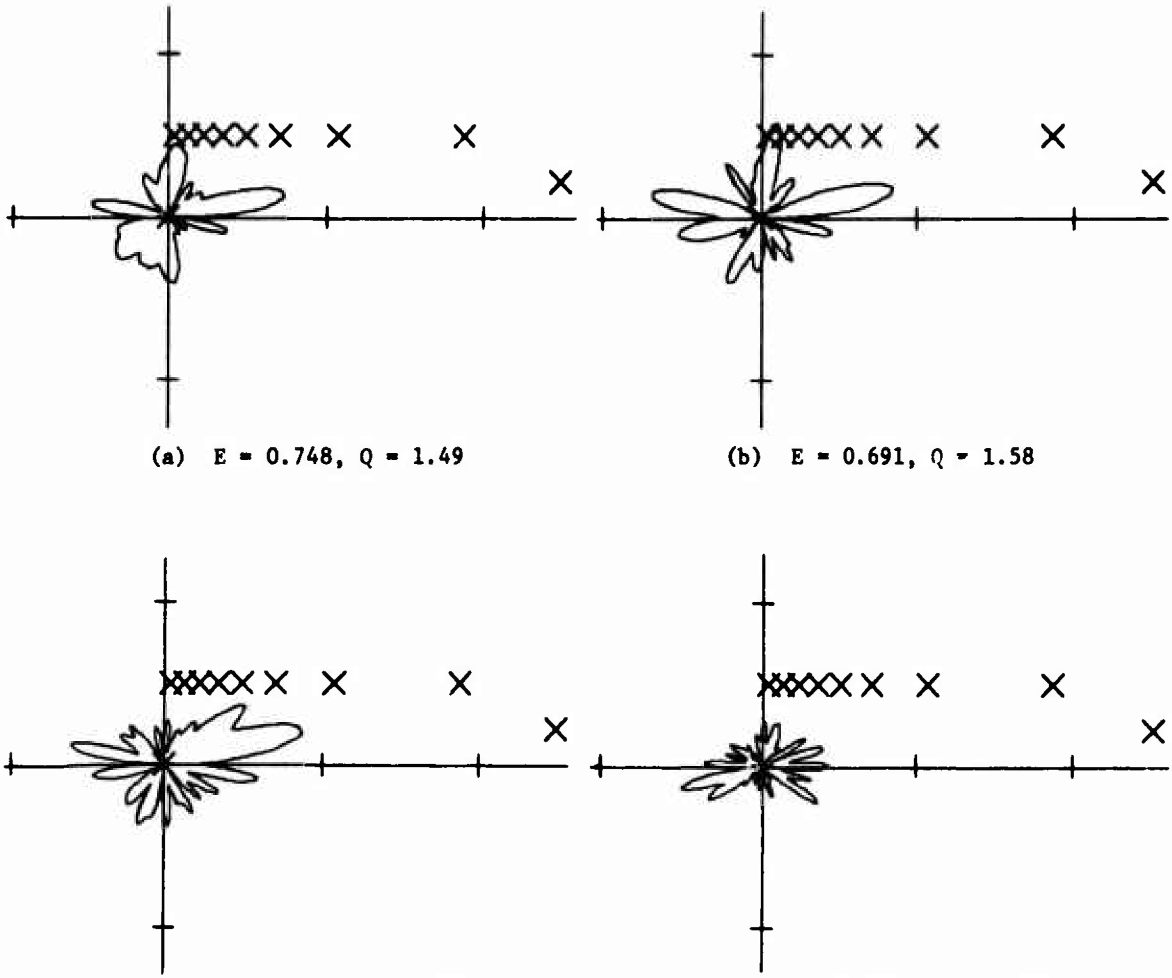


(c) E = 0.438, Q = 1.06



(d) E = 0.956, Q = 1.40

Fig. 4. Antenna pattern synthesis, phase specified, d = $\lambda/2$. Specified pattern shown by crosses, synthesized pattern shown solid.



(a) E = 0.748, Q = 1.49

(b) E = 0.691, Q = 1.58

(c) E = 0.681, Q = 1.27

(d) E = 0.872, Q = 0.889

Fig. 5. Antenna pattern synthesis, phase specified, d = λ. Specified pattern shown by crosses, synthesized pattern shown solid.

Table 1. Element excitations for the synthesized patterns of Fig. 3.

Element	Figure 3a mag.	Figure 3a phase	Figure 3b mag.	Figure 3b phase	Figure 3c mag.	Figure 3c phase	Figure 3d mag.	Figure 3d phase
f_1	0.312	-38.3°	0.164	84.2°	0.254	-29.2°	0.441	-26.7°
f_2	0.762	173.9°	0.627	-73.0°	0.657	159.2°	0.858	167.0°
f_3	1.000	0	0.810	126.4°	1.000	0	1.000	0
f_4	0.903	-147.4°	0.761	-27.8°	0.955	-147.1°	0.680	178.1°
f_5	0.502	32.6°	0.482	155.4°	0.462	55.8°	0.394	-10.3°
f_6	0.605	-122.3°	0.707	-5.4°	0.529	-137.2°	0.203	103.0°
f_7	0.507	37.1°	0.663	178.9°	0.580	43.1°	0.363	-138.6°
f_8	0.717	-126.8°	1.000	0	0.868	-147.8°	0.669	25.6°
f_9	0.647	16.4°	0.664	164.1°	0.678	17.3°	0.612	179.8°
f_{10}	0.300	149.8°	0.300	-34.3°	0.331	164.9°	0.292	-17.6°

Table 2. Element excitations for the synthesized patterns of Fig. 4.

Element	Figure 4a		Figure 4b		Figure 4c		Figure 4d	
	mag.	phase	mag.	phase	mag.	phase	mag.	phase
f_1	0.341	60.9°	0.525	-115.4°	0.383	-90.6°	0.713	-133.0°
f_2	0.315	-66.7°	0.162	-11.6°	0.340	77.5°	0.502	46.8°
f_3	0.378	22.5°	0.231	64.0°	0.297	-18.6°	0.207	167.3°
f_4	0.488	48.8°	0.248	132.9°	0.392	167.4°	0.623	65.6°
f_5	0.767	161.2°	0.661	-45.6°	1.000	0	1.000	0
f_6	0.989	-179.0°	0.939	-42.1°	0.847	-37.3°	0.783	8.6°
f_7	0.960	-112.7°	1.000	0	0.427	36.8°	0.510	57.1°
f_8	1.000	0	0.427	66.5°	0.365	136.7°	0.039	-74.5°
f_9	0.630	94.6°	0.305	-25.9°	0.341	13.8°	0.547	35.2°
f_{10}	0.057	-75.7°	0.102	130.5°	0.240	175.7°	0.734	-123.9°

Table 3. Element excitations for the synthesized patterns of Fig. 5.

Element	Figure 5a		Figure 5b		Figure 5c		Figure 5d	
	mag.	phase	mag.	phase	mag.	phase	mag.	phase
f_1	0.062	-173.5°	0.252	50.6°	0.448	-134.0°	0.619	17.3°
f_2	0.063	18.6°	0.384	-129.6°	0.284	-108.1°	0.663	21.1°
f_3	0.510	102.9°	0.467	-4.4°	0.508	-125.1°	0.134	10.1°
f_4	0.377	130.3°	0.458	-22.2°	0.379	-82.6°	1.000	0
f_5	0.497	-94.5°	0.551	120.4°	1.000	0	0.284	1.2°
f_6	0.939	153.6°	1.000	0	0.815	-150.1°	0.700	86.1°
f_7	1.000	0	0.802	168.8°	0.366	58.3°	0.150	-98.5°
f_8	0.848	-107.3°	0.442	148.5°	0.481	32.7°	0.247	26.1°
f_9	0.285	34.9°	0.129	28.5°	0.151	-64.4°	0.781	4.3°
f_{10}	0.120	-168.5°	0.068	118.9°	0.126	147.4°	0.499	2.3°

IV. FIELD MAGNITUDE PATTERN SYNTHESIS

We next consider the problem of synthesizing a pattern in magnitude only. Let $h = |g_o|$ be a desired field magnitude, and form the vector $\vec{h}$ by specifying its value h_m at M points on the radiation sphere. Again we consider the source to be discretized and represented by the vector $\vec{f}$. It is desired to find the source $\vec{f}$ for which the pattern error

$$\varepsilon = \| \, |[T]\vec{f}| - \vec{h} \|^2 \tag{19}$$

is minimum. In terms of components, (19) becomes

$$\varepsilon = \sum_{m=1}^{M} w_m \left| \, \left| \sum_{n=1}^{N} f_n T_{mn} \right| - h_m \right|^2 \tag{20}$$

where the w_m are weight factors. To circumvent the troublesome inner magnitude operation in (20), we first consider the more general function

$$\varepsilon(\vec{f},\vec{\beta}) = \sum_{m=1}^{M} w_m \left| \sum_{n=1}^{N} f_n T_{mn} - h_m e^{j\beta_m} \right|^2 \tag{21}$$

This is the error function used when the pattern is specified in both magnitude h_m and in phase β_m. Hence, for β_m fixed, the f_n for minimum ε are given by (16). For f_n fixed, the minimum ε is obtained when both terms within the magnitude signs of (21) are in phase, that is, when

$$e^{j\beta_m} = \frac{\sum_{n=1}^{N} f_n T_{mn}}{\left| \sum_{n=1}^{M} f_n T_{mn} \right|} \tag{22}$$

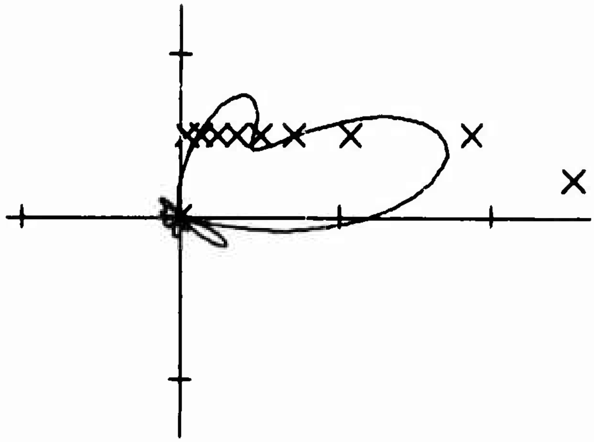
Because (21) is more general that (20), its minimum is less than or equal to that of (20). But under condition (22), the ε of (21) is equal to that of (20). Therefore (20) and (21) have the same minimum.

An iterative procedure for minimizing (21) proceeds as follows:

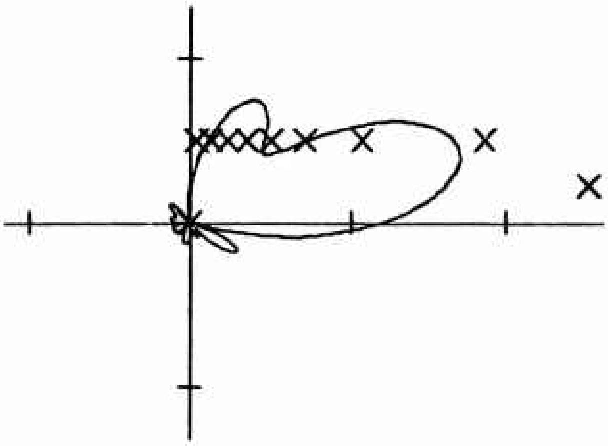
1. Assume starting values for $\beta_1, \beta_2, \ldots, \beta_M$.
2. Keep the β_m fixed and calculate the f_n which minimize ε using (16).
3. Keep the f_n fixed and calculate the β_m which minimize ε using (22).
4. Go to step 2.

This procedure eventually converges because steps 2 and 3 cannot increase ε. While the procedure obtains absolute minima in the $\vec{f}$ space and in the $\vec{\beta}$ space, it does not necessarily obtain the absolute minimum in the catenated space $(\vec{f},\vec{\beta})$. Hence, the procedure converges to a stationary point, usually a local minimum, which may or may not be the global minimum. An alternative procedure for minimizing (21) is given in the Appendix.

For numerical results, we consider the same example as used in the preceding section. Hence, the array is illustrated by Fig. 2a, and the radiation pattern by Fig. 2b. The same three cases of element separation, $d = \lambda/4$, $\lambda/2$, and λ, are used. Two starting points were chosen for the iterative procedure: (a) origin at the end point of the semimajor axis and the field real, and (b) origin at the center of the ellipse and the field phase alternating between 0 and 180° between adjacent field points. These starting points correspond to cases (c) and (d) of the previous section. The final results of the magnitude synthesis procedure are shown in Fig. 6 for $d = \lambda/4$, Fig. 7 for $d = \lambda/2$, and Fig. 8 for $d = \lambda$. The normalized synthesis errors E and quality factors Q for each result are given below each pattern. The source excitations are listed by magnitude and phase in Table 4, normalized so that the maximum excitation is unity.

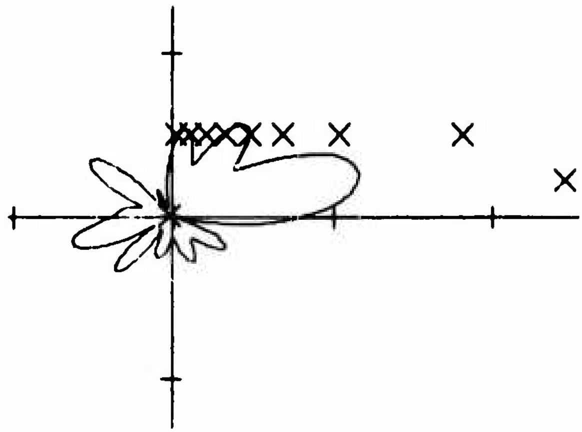


(a) E = 0.172, Q = 21.5

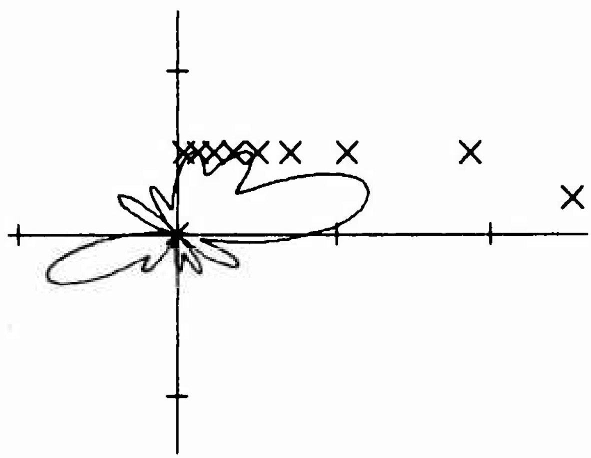


(b) E = 0.172, Q = 21.5

Fig. 6. Field magnitude pattern synthesis, d = λ/4. (a) Starting field taken real. (b) Starting field alternating 0 to 180° in phase between adjacent field points.

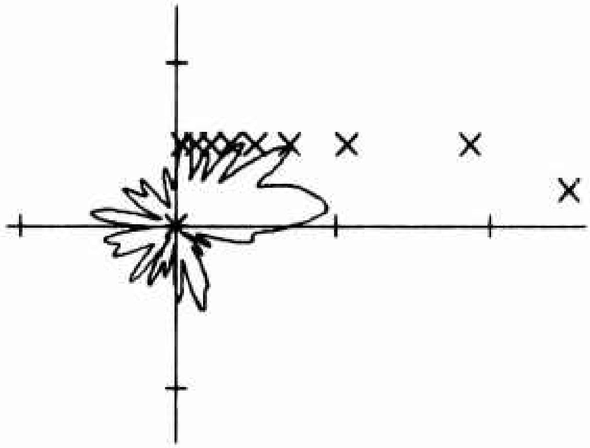


(a) E = 0.416, Q = 1.63

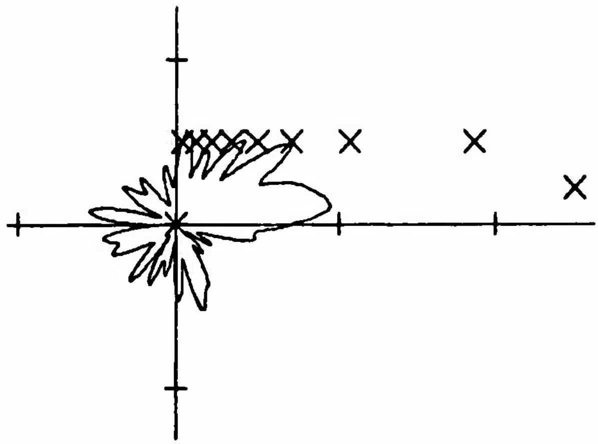


(b) E = 0.425, Q = 1.09

Fig. 7. Field magnitude pattern synthesis, d = $\lambda/2$. (a) Starting field taken real. (b) Starting field alternating 0 to 180° in phase between adjacent field points.



(a) $E = 0.517$, $Q = 0.943$



(b) $E = 0.517$, $Q = 0.943$

Fig. 8. Field magnitude pattern synthesis, $d = \lambda$. (a) Starting field taken real. (b) Starting field alternating 0 to 180° in phase between adjacent field points.

Table 4. Element excitations for the synthesized patterns of Figs. 6, 7, 8.

Element	Figure 6a		Figure 6b		Figure 7a		Figure 7b		Figure 8a		Figure 8b	
	mag.	phase	mag.	phase	mag.	phase	mag.	phase	mag.	phase	mag.	phase
f_1	0.264	125.0	0.264	125.0	0.299	-89.5	0.178	-62.0	0.461	-118.7	0.462	-118.8
f_2	0.626	-40.5	0.626	-40.5	0.362	75.4	0.337	71.2	0.552	-113.2	0.552	-113.1
f_3	0.985	153.2	0.985	153.2	0.334	-22.9	0.473	-76.9	0.680	-92.7	0.680	-92.8
f_4	1.000	0.0	1.000	0.0	0.285	-177.4	0.440	-78.5	0.970	-66.3	0.969	-66.2
f_5	0.636	-159.1	0.636	-159.2	1.000	0.0	1.000	0.0	1.000	0.0	1.000	0.0
f_6	0.553	24.7	0.553	24.7	0.827	-34.7	0.621	8.0	0.370	-66.4	0.366	-66.4
f_7	0.570	-160.3	0.570	-160.3	0.365	51.2	0.832	113.8	0.208	152.2	0.204	152.3
f_8	0.664	10.3	0.664	10.3	0.364	151.9	0.698	-154.1	0.371	75.3	0.368	75.3
f_9	0.527	172.9	0.528	172.9	0.378	12.0	0.639	-15.2	0.334	-74.7	0.334	-75.2
f_{10}	0.234	-42.8	0.235	-42.8	0.360	161.5	0.643	136.4	0.406	150.0	0.407	150.3

The following are some observations concerning the improvement of magnitude pattern synthesis over ordinary pattern synthesis. For the cases $d = \lambda/4$ and $d = \lambda/2$, when the starting field is taken real (equiphase), the improvement is relatively small. (Compare Figs. 3c to 6a, and 4c to 7a.) For the cases $d = \lambda/4$ and $d = \lambda/2$, when the starting field alternates 0 to 180° between adjacent field points, the improvement is large. (Compare Figs. 3d to 6b and 4d to 7b.) For the case $d = \lambda/4$ the two final patterns are the same (Figs. 6a and 6b), but for the case $d = \lambda/2$ the two final patterns are different (Figs. 7a and 7b). For the case $d = \lambda$, the improvement over the starting pattern is larger in both cases. (Compare Figs. 5c to 8a and 5d to 8b.) However, in no case is the final pattern very good when $d = \lambda$. Apparently the source separation is too great for good pattern synthesis. Finally, note that for the case $d = \lambda$ the two final patterns are the same (Figs. 8a and 8b).

For future reference, the source norm squared, the normalized synthesis error, and the Q for each result are tabulated in Table 5 for the case $d = \lambda/4$. The five rows correspond to the synthesized patterns of Figs. 3a, b, c, d, and Fig. 6. When the source separation is large the various quantities of Table 5 are less sensitive to the type of synthesis used, and hence we do not tabulate them for $d = \lambda/2$ and $d = \lambda$.

Table 5. Source norm squared, normalized synthesis error, and quality factor for unconstrained pattern synthesis, $d = \lambda/4$.

	$\|\|f\|\|^2$	E	Q
coordinate origin a	13.37	0.312	12.6
coordinate origin b	17.14	0.307	16.0
coordinate origin c	13.35	0.223	11.1
field phase alternating	1.38	0.957	20.5
magnitude synthesis	27.50	0.172	21.5

V. PATTERN SYNTHESIS WITH CONSTRAINED SOURCE NORM

The source norm is closely related to near-field quantities, such as power losses in an antenna structure or energy storage. For this reason it is often desirable to limit the source norm, especially when the sources are close together or continuously distributed. Hence, we consider the problem of minimizing

$$\varepsilon = \| [T]\vec{f} - \vec{g}_o \|^2 \tag{23}$$

subject to the constraint

$$\|\vec{f}\|^2 \leq C \tag{24}$$

where C is a positive constant to be chosen. This constrained minimization can be accomplished by forming the Lagrangian

$$J = \| [T]\vec{f} - \vec{g}_o \|^2 + \alpha \; \|\vec{f}\|^2 \tag{25}$$

where α is a Lagrange multiplier. If we can obtain the source function $\vec{f}$ which minimizes J with respect to $\vec{f}$ and are able to find $\alpha > 0$ such that

$$\| \vec{f} \|^2 = C \tag{26}$$

then any other source function which satisfies the constraint (24) gives at least as large a J and thus at least as large an ε as that provided by $\vec{f}$. Hence $\vec{f}$ minimizes ε subject to the constraint (24).

However, it is not always possible to find $\alpha > 0$ such that the f which minimizes J satisfies (26). This warrants an investigation into the behavior of $\|\vec{f}\|^2$ and ε attained by the minimizing function $\vec{f}$ versus α. If $\vec{f}_1$ is the minimizing function when $\alpha = \alpha_1$ and $\vec{f}_2$ is the minimizing function when $\alpha = \alpha_2$, then

$$\varepsilon_1 + \alpha_1 \ \|\vec{f}_1\|^2 \leq \varepsilon_2 + \alpha_1 \ \|\vec{f}_2\|^2 \tag{27}$$

$$\varepsilon_2 + \alpha_2 \ \|\vec{f}_2\|^2 \leq \varepsilon_1 + \alpha_2 \ \|\vec{f}_1\|^2 \tag{28}$$

where ε_1 or ε_2 is the ε of (23) with f replaced by $\vec{f}_1$ or $\vec{f}_2$. Adding the inequalities (27) and (28), we have

$$(\alpha_2 - \alpha_1)(\|f_2\|^2 - \|f_1\|^2) \leq 0 \tag{29}$$

which shows that $\|\vec{f}\|^2$ is a monotone decreasing function of α. If both α_1 and α_2 are positive, (27) and (28) can be multiplied by α_2 and α_1, respectively, without changing the sense of the inequalities. The resulting inequalities can now be added to obtain

$$(\alpha_2 - \alpha_1)(\varepsilon_2 - \varepsilon_1) \geq 0 \tag{30}$$

which shows that ε is a monotone increasing function of α for positive α. The smallest possible ε is obtained when $\alpha = 0$, corresponding to the unconstrained optimization. The smallest possible $\|\vec{f}\|^2$ is obtained when $\alpha = \infty$, because when α is very large the Lagrangian is essentially $\alpha\|\vec{f}\|^2$ and thus minimization of the Lagrangian is the same as minimization of $\|\vec{f}\|^2$.

Now if it is impossible to find $\alpha > 0$ such that the minimizing $\vec{f}$ satisfies (26), then C is either larger than $\|\vec{f}\|^2$ when $\alpha = 0$, or C is less than $\|\vec{f}\|^2$ when $\alpha = \infty$. If C is larger than $\|\vec{f}\|^2$ when $\alpha = 0$, then the desired $\vec{f}$ is the one which minimizes J when $\alpha = 0$, because this $\vec{f}$ gives the minimum possible ε and does not violate the constraint (24). Actually, the constraint (24) is ineffective whenever C is larger than $\|\vec{f}\|^2$ when $\alpha = 0$. If C is smaller than $\|\vec{f}\|^2$ when $\alpha = \infty$, then since $\|\vec{f}\|^2$ at $\alpha = \infty$ is the smallest possible $\|\vec{f}\|^2$, the constrained minimization problem (23), (24) has no solution because it is impossible to satisfy the constraint (24).

Consider the minimization of (25) for $\alpha > 0$ such that (26) is satisfied. For a fixed $\alpha > 0$, (25) has an absolute minimum because $J \geq 0$. The Lagrangian J, being a quadratic form in $\vec{f}$, has one stationary point. The absolute minimum of J must occur at this stationary point which can be obtained by setting to zero the variations of J with respect to both the real and imaginary parts of $\vec{f}$, or equivalently as on page 192 of reference [9] with respect to $\vec{f}$ and $\vec{f}^*$.

To minimize (25), expand it in terms of inner products as

$$J = \widetilde{([T]\vec{f} - \vec{g}_o)}^*[W]([T]\vec{f} - \vec{g}_o) + \alpha\tilde{\vec{f}}^*[V]\vec{f} \tag{31}$$

where [V] and [W] are the weight matrices of (8) and (11), respectively. Further expanding (31), we have

$$\begin{aligned} J = {} & \tilde{\vec{f}}^*[\tilde{T}^*WT]\vec{f} - \tilde{\vec{g}}_o^*[WT]\vec{f} - \tilde{\vec{f}}^*[\tilde{T}^*W]\vec{g}_o \\ & + \tilde{\vec{g}}_o^*[W]\vec{g}_o + \alpha\tilde{\vec{f}}^*[V]\vec{f} \end{aligned} \tag{32}$$

Taking the variation of this with respect to $\tilde{\vec{f}}^*$ and setting it equal to zero, we obtain

$$\delta J = \delta\tilde{\vec{f}}^*([\tilde{T}^*WT]\vec{f} - [\tilde{T}^*W]\vec{g}_o + \alpha[V]\vec{f}) = 0 \tag{33}$$

Since $\delta\tilde{\vec{f}}^*$ is arbitrary, the vector in the parentheses must be zero, or

$$[\tilde{T}^*WT + \alpha V]\vec{f} = [\tilde{T}^*W]\vec{g}_o \tag{34}$$

Taking the variation of (32) with respect to $\vec{f}$ and setting it equal to zero we obtain the conjugate equation to (33). Note that $\alpha = 0$ in (34)

[9] R. F. Harrington, "Field Computation by Moment Methods," Macmillan Co., New York, 1968.

gives the usual least-squares solution (equation (16) of Section III).

Next we have to determine α given the constraint $||\vec{f}||^2 = C$. For this, we first consider the eigenvalue equation

$$[\tilde{T}^* WT]\vec{\phi}_i = \lambda_i [V]\vec{\phi}_i \tag{35}$$

Let the eigenfunctions be normalized with weight [V], so the orthogonality relationships become

$$\tilde{\phi}_i^* [V]\vec{\phi}_j = \delta_{ij} \tag{36}$$

$$\tilde{\phi}_i^* [T^* WT]\vec{\phi}_j = \lambda_i \; \delta_{ij} \tag{37}$$

The matrix $[\tilde{T}^* WT]$ is Hermitian and [V] is positive definite, therefore the $\vec{\phi}_i$ form a complete set in the $\vec{f}$ space and we can write

$$\vec{f} = \sum_{i=1}^{N} \alpha_i \vec{\phi}_i \tag{38}$$

where the α_i are constants. Substituting (38) into (34), we have

$$\sum_{i=1}^{N} \alpha_i [\tilde{T}^* WT + \alpha V]\phi_i = [\tilde{T}^* W]\vec{g}_o \tag{39}$$

Premultiplying (39) by $\tilde{\phi}_j^*$, and using the orthogonality relationships (36) and (37), we obtain

$$\alpha_i (\lambda_i + \alpha) = \tilde{\phi}_i^* [\tilde{T}^* W]\vec{g}_o \tag{40}$$

which determines the α_i. Substituting these α_i into (38), we have

$$\vec{f} = \sum_{i=1}^{N} \frac{C_i}{\lambda_i + \alpha} \vec{\phi}_i \tag{41}$$

where the constants C_i are

$$\tilde{\vec{\phi}}_i^* [\tilde{T}^* W] \vec{g}_o \tag{42}$$

Next, substitute (41) into (23), again use orthogonality of the $\vec{\phi}_i$, and obtain

$$\varepsilon = \|\vec{g}_o\|^2 + \sum_{i=1}^{N} \left(\frac{-1}{\lambda_i} + \frac{\alpha^2}{\lambda_i(\lambda_i + \alpha)^2}\right)|C_i|^2 \tag{43}$$

Also, from (41) and orthogonality, we obtain

$$\|\vec{f}\|^2 = \sum_{i=1}^{N} \frac{|C_i|^2}{(\lambda_i + \alpha)^2} \tag{44}$$

Now [V] is positive definite and $[\tilde{T}^* WT]$ is at least positive indefinite, therefore all $\lambda_i \geq 0$. Hence, as expected, the error (43) is a monotone increasing function of α for $\alpha \geq 0$ and source norm squared (44) is a monotone decreasing function of α for $\alpha \geq 0$.

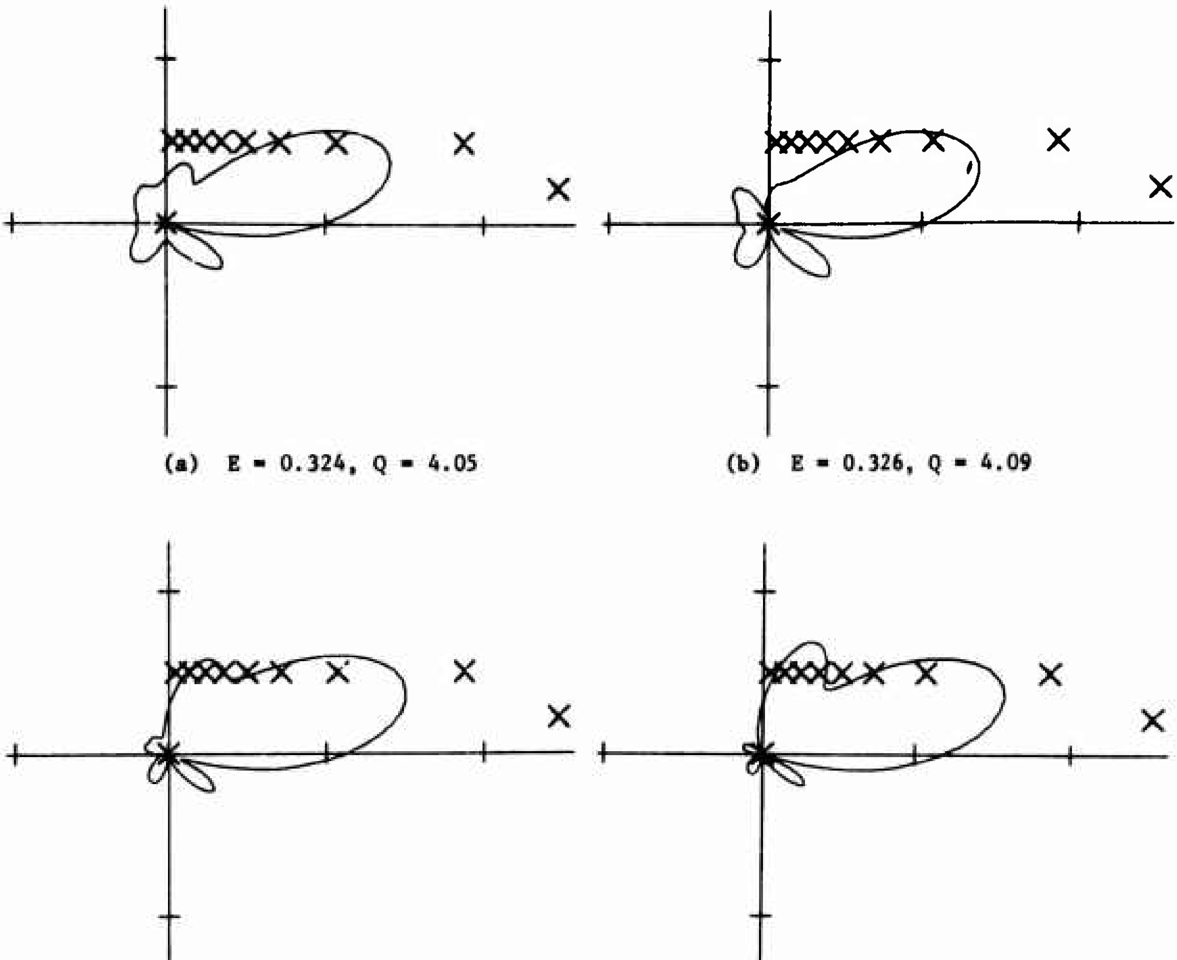
Because of the monotone decreasing nature of (44), there is precisely one $\alpha > 0$, say α_r, which satisfies

$$C - \sum_{i=1}^{N} \frac{|C_i|^2}{(\lambda_i + \alpha)^2} = F(\alpha) = 0 \tag{45}$$

This α_r can be computed using Newton's method.

The solution $\vec{f}$ which minimizes (25) for $\alpha > 0$ such that (26) is satisfied is unique and is given by (41) with $\alpha = \alpha_r$. If C is neither too large nor too small, this $\vec{f}$ minimizes ε subject to the constraint (24). As mentioned before, if C is larger than the norm squared of the $\vec{f}$ which minimizes ε when there is no constraint, then this $\vec{f}$ minimizes ε subject to the ineffective constraint (24). If C is smaller than $\|\vec{f}\|^2$ at $\alpha = \infty$, then it is impossible to satisfy the constraint (24).

For examples, the previously synthesized patterns for the case $d = \lambda/4$ were rerun with a constraint on the source norm. The resulting synthesized



(a) E = 0.324, Q = 4.05

(b) E = 0.326, Q = 4.09

(c) E = 0.235, Q = 3.55

(d) E = 0.191, Q = 6.82

Fig. 9. Pattern synthesis with constrained source norm, $d = \lambda/4$. Field phase is specified in (a), (b), and (c). Field is specified only in magnitude in (d).

patterns are shown in Fig. 9. The first three cases are for the field real with coordinate origin at the points (a), (b), and (c) of Fig. 2a. In each case the constraint was $||\vec{f}||^2 = 4$. The corresponding patterns for unconstrained synthesis are those of Figs. 3(a), (b), and (c). Note that the final synthesized patterns are not greatly different from the unconstrained results, yet $||\vec{f}||^2$ has been reduced from the order of 15 to 4 (see Table 5). The choice of coordinate origin (a) and the field phase alternating between adjacent field points was also run with the constraint $||\vec{f}||^2 = 1$, and the resulting pattern was essentially the same as Fig. 3(d). Finally, field magnitude pattern synthesis with the constraint $||\vec{f}||^2 = 8$ was run using each of the above mentioned four starting points. The final synthesized pattern was the same regardless of the starting point, and the result is shown in Fig. 9(d). Again the synthesized pattern is not greatly different from the unconstrained result, Fig. 3, even though $||\vec{f}||^2$ has been reduced from 27.5 to 8.

Table 6 lists $||\vec{f}||^2$ E, and Q for the constrained synthesis results. It should be compared to Table 5 for the corresponding unconstrained results. Note that the errors for the constrained patterns are always as high or higher than those for the unconstrained patterns, as they must be. Note also that the Q's of the constrained patterns are always as low or lower than those for the unconstrained patterns. We show in the next section that this is generally true.

Table 6. Source norm squared, normalized pattern error, and quality factor for pattern synthesis with constrained source norm.

	$\lVert\vec{f}\rVert^2$	E	Q
coordinate origin a	4.00	0.324	4.05
coordinate origin b	4.00	0.326	4.09
coordinate origin c	4.00	0.235	3.55
field phase alternating	1.00	0.957	16.15
magnitude synthesis	8.00	0.191	6.82

VI. PATTERN SYNTHESIS WITH CONSTRAINED QUALITY FACTOR

It is sometimes desirable to constrain the quality factor Q defined by (15). According to the general Lagrange multiplier theory in section V, the pattern synthesis error ε can be minimized subject to the constraint

$$Q \leq Q_o \tag{46}$$

by minimizing a Lagrangian

$$J = \varepsilon + \alpha Q \tag{47}$$

with respect to the source function $\vec{f}$ and choosing $\alpha > 0$ such that

$$Q = Q_o \tag{48}$$

at least if Q_o is neither too large nor too small.

With (23) and (15), the Lagrangian (47) is given in terms of $\vec{f}$ by

$$J = \| [T]\vec{f} - \vec{g}_o \|^2 + \alpha M \frac{\|\vec{f}\|^2}{\| [T]\vec{f} \|^2} \tag{49}$$

Similar to (32), we have

$$J = \tilde{f}^* [\tilde{T}^* W T]\vec{f} - \tilde{g}_o^* [WT]\vec{f} - \tilde{f}^* [\tilde{T}^* W]\vec{g}_o + \tilde{g}_o^* [W]\vec{g}_o + \alpha M \frac{\tilde{f}^* [V]\vec{f}}{\tilde{f}^* [\tilde{T}^* W T]\vec{f}} \tag{50}$$

Taking the variation of (50) with respect to $\tilde{f}^*$ and setting it equal to zero, we have

$$\delta J = \delta\tilde{f}^* \left(\left(1 - \frac{\alpha Q}{\| [T]\vec{f} \|^2}\right) [\tilde{T}^* W T]\vec{f} + \frac{\alpha M}{\| [T]\vec{f} \|^2} [V]\vec{f} - [\tilde{T}^* W]\vec{g}_o \right) = 0 \tag{51}$$

Since $\delta\tilde{f}^*$ is arbitrary, the vector in the parenthesis must be zero, or

$$[\tilde{T}^* WT + \frac{\alpha MV}{\|[T]\vec{f}\|^2 - \alpha Q}]\vec{f} = \frac{\|[T]\vec{f}\|^2 \; [\tilde{T}^* W]\vec{g}_o}{\|[T]\vec{f}\|^2 - \alpha Q} \tag{52}$$

Substituting (38) with $\vec{\phi}_i$ defined by (35) into (52), we obtain

$$\sum_{i=1}^{N} \alpha_i [\tilde{T}^* WT + \beta V]\vec{\phi}_i = \gamma [\tilde{T}^* W]\vec{g}_o \tag{53}$$

where

$$\beta = \frac{\alpha M}{\|[T]\vec{f}\|^2 - \alpha Q} \tag{54}$$

$$\gamma = \frac{\|[T]\vec{f}\|^2}{\|[T]\vec{f}\|^2 - \alpha Q} \tag{55}$$

Premultiplying (53) by ϕ_j^* and using the orthogonality relationships (36) and (37) we obtain

$$\alpha_i(\lambda_i + \beta) = \gamma C_i \tag{56}$$

where C_i is defined by (42). Substituting these α_i into (38), we have

$$\vec{f} = \gamma \sum_{i=1}^{N} \frac{C_i}{\lambda_i + \beta} \vec{\phi}_i \tag{57}$$

Next, using (57) and the orthogonality relationships (36) and (37), we can write the Q defined by (15) in terms β as

$$Q = M \frac{\|\vec{f}\|^2}{\|[T]\vec{f}\|^2} = M \frac{\sum_{i=1}^{N} \frac{|C_i|^2}{(\lambda_i + \beta)^2}}{\sum_{i=1}^{N} \frac{|C_i|^2 \lambda_i}{(\lambda_i + \beta)^2}} \tag{58}$$

With (57), the pattern synthesis error ε extracted from (50) is given by

$$\epsilon = \gamma\gamma^* \sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i + \beta)^2} - \gamma \sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i + \beta} - \gamma^* \sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i + \beta} + ||\vec{g}_o||^2 \quad (59)$$

Manipulation with (54), (55), and (58) results in

$$\gamma = \frac{\sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i + \beta}}{\sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i + \beta)^2}} \quad (60)$$

Equation (60) is more easily obtained by setting to zero the partial derivative of (59) with respect to γ^*. Substituting (60) into (59), we finally have

$$\epsilon = ||\vec{g}_o||^2 - \frac{\left(\sum_{i=1}^{N} \frac{|C_i|^2}{(\lambda_i + \beta)}\right)^2}{\sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i + \beta)^2}} \quad (61)$$

It is desired to find $\alpha > 0$ such that (48) is satisfied. Since (58) expresses Q not in terms of α but in terms of β, a relation between α and β must be found. Solving (54) for α, replacing Q by expression (58), and recalling that $||Tf||$ is proportional to γ given by (60), we obtain

$$\alpha = \frac{\beta}{M} \sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i + \beta} \quad (62)$$

The derivative of (62) given by

$$\frac{d\alpha}{d\beta} = \frac{1}{M} \sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i + \beta)^2} \quad (63)$$

indicates that the right hand side of (62) is a monotone increasing function of β except at $\beta = -\lambda_i$ for $i=1,2,\ldots N$ where it jumps suddenly from

$+\infty$ to $-\infty$. If the eigenvalues λ_i are arranged in decreasing order, then for a fixed $\alpha > 0$, (62) has one root β outside the interval $(-\lambda_1, 0)$. There is also a root in each of the intervals $(-\lambda_i, -\lambda_{i+1})$ for $i=1,2,\ldots N-1$. Each of these N roots represents a stationary point of the Lagrangian J. Substituting (58), (61), and (62) into (47), we obtain

$$J = \|\vec{g}_o\|^2 - \frac{\alpha M}{\beta} \tag{64}$$

This shows that the latter N-1 roots in the intervals $(-\lambda_i, -\lambda_{i+1})$ are extraneous because they all render J larger than its value at the first root outside the interval $(-\lambda_1, 0)$.

We seek β outside the interval $(-\lambda_1,0)$ such that expression (58) is equal to Q_o. Since Q is a monotone decreasing function of α and α is a monotone increasing function of β outside the interval $(-\lambda_1, 0)$, it follows that Q is a monotone decreasing function of β outside the interval $(-\lambda_1, 0)$. Hence there is only one β outside the interval $(-\lambda_1, 0)$ for which expression (58) is equal to Q_o.

That Q is a monotone decreasing function of β outside the interval $(-\lambda_1, 0)$ can be shown from (58) as follows:

$$\frac{dQ}{d\beta} = -2M \frac{\sum_{i=1}^{N} \frac{|C_i|^2}{(\lambda_i+\beta)^3} \sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i+\beta)^2} - \sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i+\beta)^3} \sum_{i=1}^{N} \frac{|C_i|^2}{(\lambda_i+\beta)^2}}{\left(\sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i+\beta)^2}\right)^2} \tag{65}$$

Replacing the product of sums in the numerator of (65) by a double sum, we can write (65) as

$$\frac{dQ}{d\beta} = -\, 2M \frac{\sum_{j=1}^{N} \sum_{i=1}^{N} \frac{|C_i|^2|C_j|^2(\lambda_i - \lambda_j)}{(\lambda_i + \beta)^2(\lambda_j + \beta)^3}}{\left(\sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i + \beta)^2}\right)^2} \tag{66}$$

Combining the (i,j) and (j,i) terms, we have

$$\frac{dQ}{d\beta} = -2M \frac{\sum_{j=2}^{N} \sum_{i=1}^{j-1} \frac{|C_i|^2 |C_j|^2 (\lambda_i - \lambda_j)^2}{(\lambda_i + \beta)^3 (\lambda_j + \beta)^3}}{\left(\sum_{i=1}^{N} \frac{|C_i|^2 \lambda_i}{(\lambda_i + \beta)^2} \right)^2} \tag{67}$$

Expression (67) is negative when β is outside the interval $(-\lambda_1, -\lambda_N)$, hence Q is a monotone decreasing function of β outside the interval $(-\lambda_1, -\lambda_N)$.

If Q_o is larger than the Q of the source function which minimizes ε when there is no constraint, namely if

$$Q_o > M \frac{\sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i^2}}{\sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i}} \tag{68}$$

then the optimum source function is the unconstrained optimum $\vec{f}$ obtained by setting $\beta = 0$ in (57) and (60). However, if Q_o is less than expression (58) at $\alpha = \infty$ corresponding to $\beta = -\lambda_1$, namely if

$$Q_o < \frac{M}{\lambda_1} \tag{69}$$

then it is impossible to satisfy the constraint (46). If

$$\frac{M}{\lambda_1} \leq Q_o \leq M \frac{\sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i^2}}{\sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i}} \tag{70}$$

then the optimum $\vec{f}$ is given by (57) and (60) where β is the unique number outside the interval $(-\lambda_1, 0)$ for which expression (58) is equal to Q_o. This β is easily computed using Newton's method with the understanding that the iterations are not allowed to proceed into the forbidden interval $(-\lambda_1, 0)$. If

$$\frac{M}{\lambda_1} \leq Q_o \leq M \frac{\sum_{i=1}^{N} |C_i|^2}{\sum_{i=1}^{N} |C_i|^2 \lambda_i} \tag{71}$$

then β is negative, but if

$$M \frac{\sum_{i=1}^{N} |C_i|^2}{\sum_{i=1}^{N} |C_i|^2 \lambda_i} < Q_o < M \frac{\sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i^2}}{\sum_{i=1}^{N} \frac{|C_i|^2}{\lambda_i}} \tag{72}$$

then β is positive.

The same examples used to illustrate the constrained $||\vec{f}||^2$ solution were run to illustrate the constrained Q solution. To check on the accuracy of the computer program, the constraint on Q was chosen to be the Q obtained for each case in Table 6. Hence, the constraint was Q = 4.05 for case (a), Q = 4.09 for case (b), Q = 3.55 for case (c), and Q = 16.15 for case (d). The final synthesized patterns were indistinguishable from those of Fig. 9. The final values of $||\vec{f}||^2$ and E are given in Table 7. Note that the error E of Table 7 is always less than or equal to that of Table 6.

Table 7. Source norm squared, normalized pattern error, and quality factor for pattern synthesis with constrained quality factor.

	$\|\|f\|\|^2$	E	Q
coordinate origin a	4.23	0.324	4.05
coordinate origin b	4.26	0.325	4.09
coordinate origin c	4.19	0.234	3.55
field phase alternating	1.08	0.957	16.15
magnitude synthesis	8.51	0.190	6.82

VII. DISCUSSION

Only a few examples are given in this report, hence it is difficult to draw general conclusions on the synthesis procedures. However, for the examples chosen, the normalized pattern synthesis error was greater than 0.4 when the sources were $\lambda/2$ or λ apart. This indicates that they cannot radiate a pattern very close to the chosen cosecant pattern. Even when the sources were $\lambda/4$ apart, the normalized pattern error was of the order of 0.3 when the phase of the radiated field was specified. The field magnitude pattern synthesis procedure reduced this error to 0.172. Constraining the source norm squared or the quality factor to about 1/3 of its unconstrained value increased this error to only 0.191. Hence, it is possible to reduce $||\vec{f}||^2$ or Q by factors greater than 3 with little change in the synthesized pattern.

The unconstrained least-squares pattern synthesis procedure gives the source vector (16). When this is substituted into (23), we obtain for the pattern synthesis error

$$\varepsilon = ||\vec{g}_o||^2 - ||[T]\vec{f}||^2 \tag{73}$$

If the pattern error is large, the synthesized pattern $[T]\vec{f}$ must have a small norm compared to that of $\vec{g}_o$. The worst possible case would be that for which the space of $[T]\vec{f}$ is orthogonal to $\vec{g}_o$. In this case the minimum synthesis error would be $||\vec{g}_o||^2$ and the source vector would be $\vec{f} = 0$.

The constrained norm pattern synthesis gives

$$||[T]\vec{f}||^2 = \sum_{i=1}^{N} \frac{|C_i|^2\lambda_i}{(\lambda_i + \alpha)^2} \tag{74}$$

which is certainly less than the norm squared of the pattern obtained from unconstrained pattern synthesis. If the constrained Q source vector is written as $\gamma\vec{f}$, then γ must be chosen to minimize

$$\varepsilon = ||\gamma\ [T]\vec{f} - \vec{g}_o||^2 \tag{75}$$

Setting to zero $\frac{d\varepsilon}{d\gamma^*}$ we obtain

$$\gamma = \frac{\tilde{\vec{f}}^*[\tilde{T}^* W]\vec{g}_o}{||T\vec{f}||^2} \tag{76}$$

$$\varepsilon(\gamma) = ||\vec{g}_o||^2 - ||\gamma[T]\vec{f}||^2 \geq 0 \tag{77}$$

so that the norm squared of the constrained Q synthesized pattern is less than or equal to $||\vec{g}_o||^2$.

The field magnitude synthesis procedure of section IV can be looked at from a different point of view. Expression (16) implicitly defines an operator, say P, which gives the phases of the elements of $\vec{f}$ in terms of the phases β_m of the elements of $\vec{g}_o$ when the magnitudes h_m of the elements of $\vec{g}_o$ are fixed. The iterative procedure of section IV successively operates with P on an initial vector $\vec{\beta}_o$ of phases.

$$\begin{aligned} \vec{\beta}_1 &= P(\vec{\beta}_o) \\ \vec{\beta}_2 &= P(\vec{\beta}_1) \\ &\vdots \\ \vec{\beta}_{n+1} &= P(\vec{\beta}_n) \end{aligned} \tag{78}$$

Convergence is obtained when $\vec{\beta}_{n+1}$ approaches $\vec{\beta}_n$. Evidently,

$$\lim_{n\to\infty} \vec{\beta}_n \tag{79}$$

is a fixed point of the operator P. The successive operations with P could have been replaced by Newton's method in which the previous iterate $\vec{\beta}_n$ is improved by adding $\Delta\vec{\beta}$

$$P(\vec{\beta}_n + \Delta\vec{\beta}) = \vec{\beta}_n + \Delta\vec{\beta}$$

$$P(\vec{\beta}_n) + \frac{\partial P}{\partial \vec{\beta}_n} \Delta\vec{\beta} = \vec{\beta}_n + \Delta\vec{\beta} \tag{80}$$

$$\left(\frac{\partial P}{\partial \vec{\beta}_n} - 1\right)\Delta\vec{\beta} = \vec{\beta}_n - P(\vec{\beta}_n)$$

Presumably, Newton's method requires fewer iterations, but each iteration is complicated by the expression $\left(\frac{\partial P}{\partial \vec{\beta}_n} - 1\right)$ which is actually a square matrix to be computed and inverted.

The eigenvectors $\vec{\phi}_i$ of (35) will be real if the elements of $[\tilde{T}^* T]$ are real. The following development reveals the circumstances under which the elements of $[\tilde{T}^* T]$ are real, or nearly real. Using (18),

$$(\tilde{T}^* T)_{mn} = \sum_{i=1}^{M} e^{jk\rho_{mn}\cos(\phi_i - \phi_o)} \tag{81}$$

where

$$\rho_{mn} = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} \tag{82}$$

$$\phi_o = \tan^{-1} \frac{y_n - y_m}{x_n - x_m} \tag{83}$$

Using the wave transformation [11]

$$e^{jk\rho_{mn}\cos(\phi_i - \phi_o)} = \sum_{q=-\infty}^{\infty} j^q \, J_q(k\rho_{mn}) e^{jq(\phi_i - \phi_o)} \tag{84}$$

one obtains

$$(\tilde{T}^* T)_{mn} = \sum_{q=-\infty}^{\infty} j^q \, J_q(k\rho_{mn}) e^{-jq\phi_o} \sum_{i=1}^{M} e^{jq\phi_i} \; . \tag{85}$$

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[11] R. F. Harrington, "Time-Harmonic Electromagnetic Fields," McGraw-Hill 1961, p. 231, Eq. (5-101).

Since the ϕ_i are equally spaced and because ϕ_1 can be absorbed into ϕ_o, we are at liberty to take

$$\phi_i = (i-1)\frac{2\pi}{M} \qquad (86)$$

in which case

$$\sum_{i=1}^{M} e^{jq\phi_i} = \begin{cases} M & q = \text{an integer multiple of } M \\ 0 & \text{otherwise} \end{cases} \qquad (87)$$

whence

$$(\tilde{T}^* T)_{mn} = M \sum_{q=-\infty}^{\infty} j^{Mq} J_{Mq}(k\rho_{mn}) e^{-jMq\phi_o} \qquad (88)$$

or

$$(\tilde{T}^* T)_{mn} = M[J_o(k\rho_{mn}) + 2 \sum_{q=1}^{\infty} j^{Mq} J_{Mq}(k\rho_{mn}) \cos(Mq\phi_o)] \qquad (89)$$

Hence $(\tilde{T}^* T)_{mn}$ is real whenever M is even. If M is odd, then $(\tilde{T}^* T)_{mn}$ is nearly real whenever $M >> k\rho_{mn}$ because from the table on page 407 of [12] $J_n(x)$ is very small when $n >> x$.

Newton's method was used to find the root α of (45) and the root β of (48) where Q is given by (58). However, if the starting value is far from the root, the first few iterates of Newton's method are probably not very well directed because the change in the variable α or β from one iteration to the next is not a good indicator. In the beginning, an interval halving procedure would probably give as much improvement per iteration and take a lot less time per iteration because the derivative need not be calculated. However, the interval halving procedure requires a negative and a positive value of the function in order to start. Also, it would be difficult to decide when to change from interval halving to Newton's method. Newton's

[12] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables," National Bureau of Standards, 1964, pp. 355-433.

method may be modified by replacing the derivative by a finite difference approximation, but then it is feared that the time saved per iteration may be offset by slower convergence.

The computer programs, with operating instructions and sample input-output data, for all the examples of this report will be given in Scientific Report No. 3 of this contract. It is hoped that further examples will be run in the future to better assess the capabilities of these synthesis programs.

APPENDIX

ALTERNATIVE METHOD FOR FIELD MAGNITUDE PATTERN SYNTHESIS

An alternative method for field magnitude synthesis is as follows. Expression (21) can be rewritten as

$$\varepsilon(\vec{f},\beta) = -\,2\,\mathrm{Real}\left(e^{-j\beta_i} w_i h_i \sum_{n=1}^{N} f_n T_{in}\right)$$

$$w_i h_i^2 + w_i \left| \sum_{n=1}^{N} f_n T_{in} \right|^2 + \sum_{\substack{m=1 \\ m\neq i}}^{M} w_m \left| \sum_{n=1}^{N} f_n T_{mn} - h_m e^{j\beta_m} \right|^2 \qquad \text{(A-1)}$$

The minimum of expression (A-1) with respect to β_i occurs when β_i is the angle of $\sum_{n=1}^{N} f_n T_{in}$. Thus it is possible to minimize (21) with respect to β_i for $i = 1,2,\ldots M$. The proposed alternative method for field magnitude synthesis consists of minimizing (21) successively with respect to $\beta_1, \beta_2, \ldots \beta_M, \beta_1, \beta_2, \ldots \beta_M$, etc. Hence, one angle is changed at a time in this method, compared to all angles being changed at once in the method used in the text. It is not known which method converges faster, since tests were not made.

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